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CONTENTS.

A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

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To prove (ii), we note that as $m = 2\rho$,

$$a_{\rho+1} = a_{\rho},$$

and $$2a_{\rho} = a_{\rho} + a_{\rho+1} \equiv 0 \pmod{k_{\rho}},$$

from (2.2). If $k_{\rho} = 2$, $f(2)$ or $f(-2)$ has solutions, but not both. If $k_{\rho} > 2$, let $k_{\rho} = \nu$ or 2ν, according as it is odd or even. In either case ν is a divisor of a_{ρ} and of $n - a_{\rho}^2$, and therefore of n. Also $\nu \leqslant a_{\rho} < \sqrt{n}$. Again, k_{ρ} is not a multiple of 4; for if it be equal to $4\nu'$, a_{ρ} is a multiple of $2\nu'$ and therefore even, making n a multiple of 4, which cannot be, as n has no square factors. Hence k_{ρ} is a t, and $f(k_{\rho})$ or $f(-k_{\rho})$ has solutions, but not both.

To prove (iii): *First* let $k_{\rho} > 2$, and equal to ν or 2ν. As ν is a divisor of n, it is a divisor of a_{ρ}. Write $a_{\rho} = c\nu$,

$$(\sqrt{n} + c\nu)(\sqrt{n} - c\nu) = k_{\rho-1}\, k_{\rho}$$

and $$(\sqrt{n} + a_{\rho+1})(\sqrt{n} - a_{\rho+1}) = k_{\rho}\, k_{\rho+1}$$

From (2.2),

$$\sqrt{n} - a_{\rho+1} = \sqrt{n} + c\nu - k_{\rho}\,[(\sqrt{n} + c\nu)/k_{\rho}]$$

or $$a_{\rho+1} = -c\nu + k_{\rho}\,[(\sqrt{n} - c\nu + 2c\nu)/k_{\rho}]$$

Now $k_{\rho} > \sqrt{n} - c\nu > 0$; and $2c\nu$ is a multiple of k_{ρ} whether k_{ρ} is ν or 2ν. Hence

$$[(\sqrt{n} - c\nu + 2c\nu)/k_{\rho}] = 2c\nu / k_{\rho}$$

and $$a_{\rho+1} = -c\nu + 2c\nu = c\nu = a_{\rho}$$

giving $$k_{\rho+1} = k_{\rho-1} \text{ etc., } k_{2\rho} = 1, \text{ and } m = 2\rho.$$

Secondly: Let $k = 2$. The same argument applies and the same conclusions follow. k_{ρ} is therefore the turning point and $m = 2\rho$ for all t.

4. There is no general method of ascertaining, without calculating the a's and the k's, which of the $4q + 3$ equations has a solution, but for particular values of n, some information,—occasionally full informa-

tion,—can be obtained by treating the equation as a congruence and by applying the theory of quadratic residues.

To take a simple illustration, let it be required to find the values of k, ($k = -1, \pm 2, \pm 3, \pm 6$), for which

$$x^2 - 15y^2 = k$$

can be solved. With the usual notation for the quadratic residue character of a number with respect to another, we have

$$(k/3) = +1 \text{ and } (k/5) = +1$$

excluding $k = -1, \pm 2, \pm 3$. If k is even, but not divisible by 4, x and y are odd, and

$$k \equiv x^2 + y^2 \equiv 2 \pmod{8}$$

excluding $x = 6$. The remaining number, -6, must furnish solutions.

5. The method of treating the equation as a congruence gives the following results when n is odd, and $k = -1, 2$, or -2;

$$\left.\begin{array}{ll} \text{(i)} & \text{If } f(-1) \text{ has solutions, } n \equiv 1, \text{ or } 5\ ; \\ \text{(ii)} & \text{If } f(2) \text{ has solutions, } n \equiv 7\ ; \\ \text{(iii)} & \text{If } f(-2) \text{ has solutions, } n \equiv 3 \end{array}\right\} \pmod{8}$$

these conditions being necessary, but not sufficient. If however n is a prime, it has no factors $< \sqrt{n}$, and the $4q + 3$ equations of § 3 reduce to $f(-1)$, $f(2)$, $f(-2)$. As one of these must have a solution, the above conditions are also sufficient.

If n is not a prime, $f(k)$ is possible only if k is a quadratic residue of every odd prime factor of n, and n is a quadratic residue of every odd prime factor of k. Confining ourselves to $k = -1, 2, -2$, we see that

(i) If $f(-1)$ has solutions, every odd factor of n is of the form $4p + 1$;

(ii) If $f(2)$ has solutions, every odd factor of n is of the form $8p \pm 1$;

and (iii) If $f(-2)$ has solutions, every odd factor of n is of the form $8p + 1$ or $8p + 3$.

These conditions are however not sufficient.

6. We have somewhat similar results if $n = 2N$;

$$\left.\begin{array}{lll} & \text{from } x^2 - 2Ny^2 = -1, & \text{we get } N \equiv 1 \text{ or } 5; \\ & \text{from } x^2 - 2Ny^2 = 2, & \text{we get } N \equiv 1 \text{ or } 7 \\ \text{and} & \text{from } x^2 - 2Ny^2 = -2, & \text{we get } N \equiv 1 \text{ or } 3 \end{array}\right\} \pmod{8}$$

as conditions that are necessary but not sufficient in general. If however N is prime, $n = 2N$ has again no odd factors $< \sqrt{n}$, and one out of $f(-1)$, $f(2)$ and $f(-2)$ must have a solution. The above conditions are now sufficient, except when $N \equiv 1 \pmod{8}$, which case is not settled by this analysis.

7. If $N \equiv 1 \pmod{8}$, we have

$$\left.\begin{array}{l} N = \alpha^2 + 16\beta^2 \\ \quad = c^2 + 8d^2 \\ \quad = 2g^2 - h^2 \end{array}\right\} \quad \left.\begin{array}{l} n = 2N = (\alpha + 4\beta)^2 + (\alpha - 4\beta)^2 \\ \qquad = (4d)^2 + 2c^2 \\ \qquad = (2g)^2 - 2h^2 \end{array}\right\} \quad \ldots \quad (7.1)$$

where α, c, g, h are odd, and α, β, c d, are unique, while g and h have a limited range of values. In all cases, we may write

$$n = A^2 - kB^2,$$

where $k = -1$, -2 or 2, and B is odd.

We further re-call the fact that every factor of

$$P^2 - kQ^2,$$

(where P and Q are prime to each other, and $k = -1$, -2 or 2) can be expressed in the same form in one or more ways.

8. Let the solution of $f(k)$, where $k = -1$, -2, or 2 be given by x and y, so that

$$x^2 - ny^2 = k,$$

x and y being prime to each other. As y is a factor of $x^2 - k$, we may put

$$y = \lambda^2 - k\mu^2 = (\lambda + \sqrt{k}.\mu)(\lambda - \sqrt{k}.\mu)$$

where (λ, μ) may have more than one set of values, and may be positive or negative.

$$x^2-k=ny^2 = (A^2-kB^2)(\lambda^2-k\mu^2)^2$$
$$= (A+B\sqrt{k})(\lambda+\mu\sqrt{k})^2(A-B\sqrt{k})(\lambda-\mu\sqrt{k})$$
$$= (A+B\sqrt{k})(\lambda^2+k\mu^2+2\lambda\mu\sqrt{k}) \times \text{the conjugate factor}$$
$$= [A(\lambda^2+k\mu^2)+2B\lambda\mu k]^2-k[B(\lambda^2+k\mu^2)+2A\lambda\mu]^2$$
$$= X^2-kY^2 \text{ (say)} \qquad \ldots \quad (8.1)$$

There will, in general, be several sets of values of (X, Y) according to the values and the signs of A, B, λ, μ; but as (8.1) is an identity, we must have at least once $Y^2 = 1$.

For this value of y

$$B(\lambda^2+k\mu^2)+2A\lambda\mu = \pm 1$$

or

$$(B\lambda+A\mu)^2-(A^2-kB^2)\mu^2 = \pm B$$

or

$$\xi^2-n\eta^2 = \pm B \qquad \ldots \quad (8.2)$$

As B is odd, ξ is odd; $\xi^2 \equiv 1$ (mod. 8). Also $n \equiv 2$, (mod. 8); it follows that

$$B = \pm[\xi^2-n\eta^2] \equiv \pm(1-2\eta^2) = \pm 1, \qquad \ldots \quad (8.3)$$

whether η is odd or even. If b be a prime factor of B, (8.2) requires that $(2N/b) = +1$, and $(\pm B/N) = 1$. The reader may prove, or verify, that these conditions also reduce to (8.3).

For $f(k)$ to have solutions, it is necessary but not sufficient that (8.3) should be satisfied. In other words $f(k)$ has no solutions if $B \equiv \pm 3$ (mod. 8).

9. This can be translated in terms of α and β.

(i) When $k=-1$, $B=\alpha+4\beta$ or $\alpha-4\beta$; $B \equiv \pm 3$ is equivalent to

$$\alpha \equiv \pm 1 \text{ with } \beta \text{ even}$$

or

$$\alpha \equiv \pm 3 \text{ with } \beta \text{ odd}$$

(the modulus being 8). $f(-1)$ has no solutions in these cases, but may have solutions otherwise. ... (9.1)

(ii) when $k=-2$, $B=c$; and $\alpha^2-8d^2=c^2-16\beta^2$.

Every factor of α^2-8d^2 is of the form $8p\pm1$; therefore $c+4\beta$ is of that form. If $c \equiv \pm 3$ (mod. 8), β is odd.

Hence $f(-2)$ has no solutions if β is odd, but may have solutions if β is even. ... (9.2)

(iii) When $k = 2$, $B = h$; and

$$\alpha^2 + h^2 = 2g^2 + 16\beta^2 \equiv 2 \pmod{16}, \text{ as } g \text{ is odd}$$

i.e. $\alpha \equiv \pm 1 \pmod 8$, if h is of that form,

and $\alpha \equiv \pm 3 \pmod 8$, if h is of that form.

Hence $f(2)$ has no solutions if $\alpha \equiv \pm 3 \pmod 8$, but may have solutions otherwise. ... (9.3)

10. Putting (9.1), (9.2) and (9.3) together and remembering that one of the three equations must have a solution, we obtain the following

THEOREM: Of the three equations

$$x^2 - 2Ny^2 = k,$$

where $k = -1$, -2, or 2, and N is a prime number of the form $8p + 1$, and therefore equal to $\alpha^2 + 16\beta^2$,

(i) $k = -1$ alone gives solutions if $\alpha = 8j \pm 3$ and β is odd,

(ii) $k = -2$ alone gives solutions if $\alpha = 8j \pm 3$ and β is even:

(iii) $k = 2$ alone gives solutions if $\alpha = 8j \pm 1$ and β is odd,

and any one, but not more than one, may have solutions if $\alpha = 8j \pm 1$, and β is even.

11. I find, by actual calculation, that for all primes of the form

$$N = (8j \pm 1)^2 + 64\gamma^2$$

up to $N = 3761$,

(i) $f(-1)$ has solutions for

N = 113, 1201, 1601, 1777, 2113, 3089, 3121, 3137, 3313, 3761;

(ii) $f(-2)$ has solutions for

N = 257, 1153, 1217, 1553, 2593, 2657, 2689, 2833;

(iii) $f(2)$ has solutions for

N = 337, 353, 577, 593, 881, 1249, 1889, 2129, 3217, 3361.

Miquel points and Circles and Centre-circles of a system of lines*

By V. Ramaswami Aiyar and M. Bhimasena Rao.

1. Introduction.—Let us consider a system of co-planar lines which are such that no two of them are parallel, and no three concurrent. Taking four such lines, we know that the circum-circles of the four triangles which they form, are concurrent at a point (M_4), called the *Miquel point* of the lines. We also know that the centres of these circum-circles lie on a circle (C_4) which is called the *centre circle* of the lines. These theorems stand at the apex of two known series of theorems †, one relating to Miquel points and circles, and the other to centre circles of n lines, $(n > 4)$. The first of the series may be enunciated thus :—

(*a*) With any odd number n of lines, $(n = 5, 7, \dots)$ there is associated a circle, called the *Miquel circle* (M_n) of the lines, which passes through the n Miquel points (M_{n-1}) of the lines taken in sets of $n-1$.

(*b*) With any even number n of lines, $(n = 6, 8, \dots)$ there is associated a point (M_n), called the *Miquel point* of the lines through which pass the n Miquel circles of the lines taken in sets of $n-1$.

The second series of theorems may be stated thus :—

Given n lines $(n = 5, 6, 7, \dots)$ the n centre circles (C_{n-1}) of these lines taken in sets of $n-1$ at a time are concurrent at a point (P_n); and further the centres of these n centre circles (C_{n-1}) lie on a circle (C_n), called the *centre circle* of the lines.

2. These theorems are proved here analytically in a new manner, in which the circles are expressed in the form of determinants, and the points in the form of matrices.

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* Read at the Fifth Conference of the Indian Mathematical Society held at Bangalore in April 1926.

† *Vide* Coolidge : ***Treatise on the Circle and the Sphere*** (1916), pp. 90–92.

The following principle in the theory of determinants is chiefly employed :—

If, in a matrix containing n columns, and $n+1$ rows two of the determinants of the nth order contained in the matrix vanish, then all the determinants of the nth order contained in the matrix vanish,—provided that in the matrix (of n columns and $n-1$ rows) which is common to the determinants, the contained determinants of the $n-1$th order do not all vanish. It is hardly necessary to add that in this statement we may interchange the words "rows" and "columns."

3. The circum-circle of three lines and the Miquel point of four lines.

Referred to a triangle ABC, let the lines of the system be

$$L_r \equiv l_r\alpha + m_r\beta + n_r\gamma = 0. \quad (r = 1, 2, 3 \ldots)$$

Let the result of substituting the co-ordinates of the circular points I, J in L_r be denoted by x_r, y_r.

Taking the lines L_1, L_2, L_3, the equation of the circum-circle of the triangle formed by them is at once written in the form

$$\begin{vmatrix} \frac{L_1}{x_1} & \frac{L_1}{y_1} & 1 \\ \frac{L_2}{x_2} & \frac{L_2}{y_2} & 1 \\ \frac{L_3}{x_3} & \frac{L_3}{y_3} & 1 \end{vmatrix} = 0 \qquad \ldots (3.1)$$

for, this represents a conic which passes through the intersection of the lines as well as through the circular points I (x_1, x_2, x_3) and J (y_1, y_2, y_3).

Similarly the circumcircle of the triangle formed by the lines L_1, L_2, L_4 is

$$\begin{vmatrix} \frac{L_1}{x_1} & \frac{L_1}{y_1} & 1 \\ \frac{L_2}{x_2} & \frac{L_2}{y_2} & 1 \\ \frac{L_4}{x_4} & \frac{L_4}{y_4} & 1 \end{vmatrix} = 0 \qquad \ldots (3.2)$$

These circles intersect in two points one of which is $(L_1 L_2)$. Denote the other by M_4. The determinants (3.1) and (3.2) vanish for both these points.

Consider now the matrix common to (3.1) and (3.2), *viz.*

$$\left\| \begin{array}{ccc} \frac{L_1}{x_1} & \frac{L_1}{y_1} & 1 \\ \frac{L_2}{x_2} & \frac{L_2}{y_2} & 1 \end{array} \right\| \quad \ldots \quad \ldots \quad (3.3)$$

The determinants of the second order contained in this matrix vanish for the point (L_1L_2) and for that point only, and do not vanish for the point M_4. Hence by the principle in § 2, all the determinants of the third order contained in the matrix :—

$$\left\| \begin{array}{ccc} \frac{L_1}{x_1} & \frac{L_1}{y_1} & 1 \\ \frac{L_2}{x_2} & \frac{L_2}{y_2} & 1 \\ \frac{L_3}{x_3} & \frac{L_3}{y_3} & 1 \\ \frac{L_4}{x_4} & \frac{L_4}{y_4} & 1 \end{array} \right\| \quad \ldots \quad (3.4)$$

vanish for the point M_4, showing that the circum-circles of all the four triangles formed by L_1, L_2, L_3, L_4 are concurrent at M_4.

4. Miquel circle for five lines and Miquel point for six lines.

Taking the matrix (3.4) which represents the Miquel point of four lines L_1, L_2, L_3, L_4 and dividing the rows by x_1, x_2, x_3, x_4 we have the matrix

$$\left\| \begin{array}{ccc} \frac{L_1}{x_1^2} & \frac{L_1}{x_1 y_1} & \frac{1}{x_1} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \frac{L_4}{x_4^2} & \frac{L_4}{x_4 y_4} & \frac{1}{x_4} \end{array} \right\| \quad \ldots \quad (4.1)$$

whose determinants of the third order all vanish for the Miquel point M_4. Hence the determinant

$$\begin{vmatrix} \frac{L_1}{x_1^2} & \frac{L_1}{x_1 y_1} & \frac{1}{x_1} & \frac{1}{y_1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{L_4}{x_4^2} & \frac{L_4}{x_4 y_4} & \frac{1}{x_4} & \frac{1}{y_4} \end{vmatrix} \quad \ldots \quad (4.2)$$

vanishes for M_4, as is obvious by expanding the determinant in terms of the elements of the last column. Again taking the matrix (3.4) and dividing the rows by y_1, y_2, y_3, y_4, we have the matrix

$$\begin{Vmatrix} \frac{L_1}{x_1 y_1} & \frac{L_1}{y_1^2} & \frac{1}{y_1} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \frac{L_4}{x_4 y_4} & \frac{L_4}{y_4^2} & \frac{1}{y_4} \end{Vmatrix} \quad \ldots \quad (4.3)$$

whose determinants of the third order vanish for M_4. Hence the determinant

$$\begin{vmatrix} \frac{L_1}{x_1 y_1} & \frac{L_1}{y_1^2} & \frac{1}{x_1} & \frac{1}{y_1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{L_4}{x_4 y_4} & \frac{L_4}{y_4^2} & \frac{1}{x_4} & \frac{1}{y_4} \end{vmatrix} \quad \ldots \quad (4.4)$$

vanishes for M_4 as is obvious by expanding in terms of the elements of the third column.

From (4.2) and (4.4) we conclude by means of the principle in § 2, that all the determinants of the fourth order contained in the matrix

$$\left\|\begin{matrix} \frac{L_1}{x_1^2} & \frac{L_1}{x_1y_1} & \frac{L_1}{y_1^2} & \frac{1}{x_1} & \frac{1}{y_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{L_4}{x_4^2} & \frac{L_4}{x_4y_4} & \frac{L_4}{y_4^2} & \frac{1}{x_4} & \frac{1}{y_4} \end{matrix}\right\| \quad \ldots \quad (4.5)$$

vanish for the point M_4.

To justify this conclusion, we have to be sure that not all the determinants of the common matrix of (4.2) and (4.4), namely of,

$$\left\|\begin{matrix} \frac{L_1}{x_1y_1} & \frac{1}{x_1} & \frac{1}{y_1} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \frac{L_4}{x_4y_4} & \frac{1}{x_4} & \frac{1}{y_4} \end{matrix}\right\| \quad \ldots \quad (4.6)$$

vanish for the point M_4. The determinants in this matrix all denote the line at infinity, and hence do not vanish for the point M_4.

The original matrix (3.4) for the Miquel point M_4 has thus been transformed into a new matrix (4.5) denoting the same point. Such a transformation, it will be seen, occurs at each stage of our demonstration.

5. Take now a fifth line L_5 and consider the determinant

$$\left|\begin{matrix} \frac{L_1}{x_1^2} & \frac{L_1}{x_1y_1} & \frac{L_1}{y_1^2} & \frac{1}{x_1} & \frac{1}{y_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{L_5}{x_5^2} & \frac{L_5}{x_5y_5} & \frac{L_5}{y_5^2} & \frac{1}{x_5} & \frac{1}{y_5} \end{matrix}\right| = 0 \quad (5.1)$$

Expanding in terms of the elements of the last row, we see that the determinant (5.1) represents a locus passing through the Miquel point M_4 of the lines L_1, L_2, L_3, L_4. In like manner, the locus passes through all the Miquel points of L_1, L_2, L_3, L_4, L_5 taken in sets of four, The locus represented by (5.1) is apparently a cubic. But it is shown below that it breaks up into the line at infinity and a circle; consequently we infer that this circle, call it M_5, is one passing through the five Miquel points of the lines L_1, L_2, L_3, L_4, L_5, taken in sets of four. It is the Miquel circle of the lines.

To show that (5.1) breaks into the line at infinity and a circle, let the triangle of reference be AIJ, where I, J are the circular points. The co-ordinates of I, J can now be taken as (0, 1, 0) and (0, 0, 1). The result of substituting the co-ordinates of I, J in $L_r \equiv l_r\alpha + m_r\beta + n_r\gamma$, is m_r, n_r respectively. Hence $x_r = m_r$, $y_r = n_r$, and (5 1) takes the form

$$\begin{vmatrix} \dfrac{l_1\alpha + m_1\beta + n_1\gamma}{m_1^2}, & \dfrac{l_1\alpha + m_1\beta + n_1\gamma}{m_1 n_1}, & \dfrac{l_1\alpha + m_1\beta + n_1\gamma}{n_1^2}, & \dfrac{1}{m_1}, & \dfrac{1}{n_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{l_5\alpha + m_5\beta + n_5\gamma}{m_5^2}, & \dfrac{l_5\alpha + m_5\beta + n_5\gamma}{m_5 n_5}, & \dfrac{l_5\alpha + m_5\beta + n_5\gamma}{n_5^2}, & \dfrac{1}{m_5}, & \dfrac{1}{n_5} \end{vmatrix} \quad \ldots (5.2)$$

which easily reduces to

$$\begin{vmatrix} \dfrac{l_1\alpha + n_1\gamma}{m_1^2}, & \dfrac{l_1\alpha}{m_1 n_1}, & \dfrac{l_1\alpha + m_1\beta}{n_1^2}, & \dfrac{1}{m_1}, & \dfrac{1}{n_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{l_5\alpha + n_5\gamma}{m_5^2}, & \dfrac{l_5\alpha}{m_5 n_5}, & \dfrac{l_5\alpha + m_5\beta}{n_5^2}, & \dfrac{1}{m_5}, & \dfrac{1}{n_5} \end{vmatrix} \quad \ldots (5.3)$$

Thus α is a factor of the determinant, that is, the line at infinity is part of the locus. Taking α out, the resulting determinant which is of the second degree vanishes for $\alpha = 0$, $\gamma = 0$; (that is, for I) as well as $\alpha = 0$, $\beta = 0$ (that is, for J). Thus the locus (5.1) is shown to consist of the line at infinity and a circle M_5.

6. Taking a sixth line L_6, we can now deduce from (5.1) that the six Miquel circles of the lines taken in sets of five are concurrent at a point for which all the determinants of the fifth order contained in the matrix :—

$$\left\| \begin{matrix} \frac{L_1}{x_1^2} & \frac{L_1}{x_1 y_1} & \frac{L_1}{y_1^2} & \frac{1}{x_1} & \frac{1}{y_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{L_6}{x_6^2} & \frac{L_6}{x_6 y_6} & \frac{L_6}{y_6^2} & \frac{1}{x_6} & \frac{1}{y_6} \end{matrix} \right\| \quad \ldots \quad (6.1)$$

vanish. This matrix therefore represents the Miquel point, M_6, of the six lines.

7. Miquel circle of seven lines and Miquel point of eight lines.

To obtain the Miquel circle of seven lines, we first transform the matrix (6 1) into the matrix

$$\left\| \begin{matrix} \frac{L_1}{x_1^3} & \frac{L_1}{x_1^2 y_1} & \frac{L_1}{x_1 y_1^2} & \frac{L_1}{y_1^3} & \frac{1}{x_1^2} & \frac{1}{x_1 y_1} & \frac{1}{y_1^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{L_6}{x_6^3} & \frac{L_6}{x_6^2 y_6} & \frac{L_6}{x_6 y_6^2} & \frac{L_6}{y_6^3} & \frac{1}{x_6^2} & \frac{1}{x_6 y_6} & \frac{1}{y_6^2} \end{matrix} \right\| \quad (7.1)$$

by proceeding as follows :—

We first divide the rows in (6.1) by $x_1, x_2, \ldots x_6$ respectively and adding a column whose elements are

$$\frac{1}{y_1^2}, \quad \frac{1}{y_2^2}, \quad \ldots\ldots \quad \frac{1}{y_6^2},$$

we obtain the determinant

$$\left| \begin{matrix} \frac{L_1}{x_1^3} & \frac{L_1}{x_1^2 y_1} & \frac{L_1}{x_1 y_1^2} & \frac{1}{x_1^2} & \frac{1}{x_1 y_1} & \frac{1}{y_1^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{L_6}{x_6^3} & \frac{L_6}{x_6^2 y_6} & \frac{L_6}{x_6 y_6^2} & \frac{1}{x_6^2} & \frac{1}{x_6 y_6} & \frac{1}{y_6^2} \end{matrix} \right| \quad (7.2)$$

which vanishes for the point M_6 in question. Then we divide the rows of (6.1) again by $y_1, y_2, \dots \dots y_6$ respectively, and add a column whose elements are

$$\frac{1}{x_1^2}, \frac{1}{x_2^2}, \dots \dots \frac{1}{x_6^2}$$

obtaining the determinant

$$\begin{vmatrix} \frac{L_1}{x_1^2 y_1}, & \frac{L_1}{x_1 y_1^2}, & \frac{L_1}{y_1^3}, & \frac{1}{x_1^2}, & \frac{1}{x_1 y_1}, & \frac{1}{y_1^2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{L_6}{x_6^2 y_6}, & \frac{L_6}{x_6 y_6^2}, & \frac{L_6}{y_6^3}, & \frac{1}{x_6^2}, & \frac{1}{x_6 y_6}, & \frac{1}{y_6^2} \end{vmatrix} \quad (7.3)$$

which also vanishes for the Miquel point M_6. From (7.2) and (7.3), using the principle stated in § 2, we obtain the form (7.1) for the Miquel point M_6.

Taking now a seventh line L_7, we now see that the seven Miquel points of the lines $L_1, L_2, \dots \dots L_7$ taken six at a time lie on the curve

$$\begin{vmatrix} \frac{L_1}{x_1^3}, & \frac{L_1}{x_1^2 y_1}, & \frac{L_1}{x_1 y_1^2}, & \frac{L_1}{y_1^3}, & \frac{1}{x_1^2}, & \frac{1}{x_1 y_1}, & \frac{1}{y_1^2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{L_7}{x^3}, & \frac{L_7}{x_7^2 y_7}, & \frac{L_7}{x_7 y_7^2}, & \frac{L_7}{y_7^3}, & \frac{1}{x_7^2}, & \frac{1}{x_7 y_7}, & \frac{1}{y_7^2} \end{vmatrix} = 0 \quad \dots(7.4)$$

By referring to the triangle AIJ, the above is seen to be the line at infinity taken twice and a circle M_7, which passes through the Miquel points of the seven lines taken in sets of six, and is therefore the Miquel circle of the seven lines.

8. Taking now an eighth line, L_8, we infer that the eight Miquel circles of the lines taken seven at a time are concurrent at a point M_8 for which all the determinants of the seventh order contained in the matrix

$$\left\| \begin{matrix} \frac{L_1}{x_1^3} & \frac{L_1}{x_1^2 y_1} & \frac{L_1}{x_1 y_1^2} & \frac{L_1}{y_1^3} & \frac{1}{x_1^2} & \frac{1}{x_1 y_1} & \frac{1}{y_1^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{L_8}{x_8^3}, & \frac{L_8}{x_8^2 y_8} & \frac{L_8}{x_8 y_8^2} & \frac{L_8}{y_8^3} & \frac{1}{x_8^2} & \frac{1}{x_8 y_8} & \frac{1}{y_8^2} \end{matrix} \right\| \quad \ldots (8.1)$$

vanish. Hence the matrix represents the Miquel point M_8.

9. With additional lines we proceed similarly. The Miquel point of $2n$ lines $L_1, L_2, \ldots \ldots L_{2n}$ will be represented by the matrix.

$$\left\| \begin{matrix} \frac{L_1}{x_1^{n-1}} & \frac{L_1}{x_1^{n-2} y_1} & \cdots & \frac{L}{y_1^{n-1}} & \frac{1}{x_1^{n-2}} & \frac{1}{x_1^{n-3} y_1} & \cdots & \frac{1}{y_1^{n-2}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{L_{2n}}{x_{2n}^{n-1}} & \cdots & \cdots & \frac{L_{2n}}{y_{2n}^{n-1}} & \frac{1}{x_{2n}^{n-2}} & \cdots & \cdots & \frac{1}{y_{2n}^{n-2}} \end{matrix} \right\|$$

$$\ldots (9.1)$$

containing $2n$ rows and $2n-1$ columns. This matrix is transformed into the matrix

$$\left\| \begin{matrix} \frac{L_1}{x_1^n} & \frac{L_1}{x_1^{n-1} y_1} & \cdots & \frac{L_1}{y_1^n} & \frac{1}{x_1^{n-1}} & \cdots & \frac{1}{y_1^{n-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix} \right\| \quad \ldots (9.2)$$

containing $2n$ rows and $2n+1$ columns.

The Miquel circle of $2n+1$ lines is given by

$$\left| \begin{matrix} \frac{L_1}{x_1^n}, & \frac{L_1}{x_1^{n-1} y_1}, & \frac{L_1}{x_1^{n-2} y_1^2}, & \cdots & \frac{L_1}{y_1^n} & \frac{1}{x_1^{n-1}}, & \cdots & \frac{1}{y_1^{n-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix} \right| = 0 \quad \ldots (9.3)$$

this equation representing the Miquel circle M_{2n+1} and in addition the line at infinity repeated $n-1$ times.

THE CENTRE CIRCLE.

10. The centre of the circum-circle of three lines.

We saw that the circum-circle of the triangle formed by three lines L_1, L_2, L_3 was

$$\begin{vmatrix} \frac{L_1}{x_1} & \frac{L_1}{y_1} & 1 \\ \frac{L_2}{x_2} & \frac{L_2}{y_2} & 1 \\ \frac{L_3}{x_3} & \frac{L_3}{y_3} & 1 \end{vmatrix} = 0 \quad \ldots \quad (10.1)$$

The centre of the circle is the point of intersection of the tangents to it at I and J.

The tangent at I (x_1, x_2, x_3) is seen to be

$$\begin{vmatrix} \frac{L_1}{x_1^2} & \frac{1}{x_1} & \frac{1}{y_1} \\ \frac{L_2}{x_2^2} & \frac{1}{x_2} & \frac{1}{y_2} \\ \frac{L_3}{x_3^2} & \frac{1}{x_3} & \frac{1}{y_3} \end{vmatrix} = 0 \quad \ldots \quad (10.2)$$

Similarly the tangent at J is

$$\begin{vmatrix} \frac{L_1}{y_1^2} & \frac{1}{x_1} & \frac{1}{y_1} \\ \frac{L_2}{y_2^2} & \frac{1}{x_2} & \frac{1}{y_2} \\ \frac{L_3}{y_3^2} & \frac{1}{x_3} & \frac{1}{y_3} \end{vmatrix} = 0 \quad \ldots \quad (10.3)$$

Hence for the point of intersection of these tangents, that is for the circumcentre, say O_3, all the determinants of the third order contained in the matrix

$$\begin{Vmatrix} \frac{L_1}{x_1^2} & \frac{L_1}{y_1^2} & \frac{1}{x_1} & \frac{1}{y_1} \\ \frac{L_2}{x_2^2} & \frac{L_2}{y_2^2} & \frac{1}{x_2} & \frac{1}{y_2} \\ \frac{L_3}{x_3^2} & \frac{L_3}{y_3^2} & \frac{1}{x_3} & \frac{1}{y_3} \end{Vmatrix} \quad \ldots \quad (10.4)$$

vanish. The matrix thererfore represents the circum-centre O_3.

The centre=circle of four lines.

Taking now a fourth line L_4, we see from (10.4) that the determinant

$$\begin{vmatrix} \frac{L_1}{x_1^2} & \frac{L_1}{y_1^2} & \frac{1}{x_1} & \frac{1}{y_1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{L_4}{x_4^2} & \frac{L_4}{y_4^2} & \frac{1}{x^4} & \frac{1}{y_4} \end{vmatrix} \quad \ldots \quad (10.5)$$

represents a locus passing through the circum-centre of the triangle formed by the lines L_1, L_2, L_3 as is obvious by expanding in terms of the elements of the last row. In like manner the locus passes through the circum-centres of all the triangles formed by the lines L_1, L_2, L_3, L_4. The locus also passes through I and J, and is of the second degree. It therefore denotes the centre-circle C_4 of the lines L_1, L_2, L_3, L_4.

11. Matrix for P_5.

Taking a fifth line L_5, the equation of the centre-circle of the lines L_1, L_2, L_3, L_5 will be

$$\begin{vmatrix} \frac{L_1}{x_1^2} & \frac{L_1}{y_1^2} & \frac{1}{x_1} & \frac{1}{y_1} \\ \frac{L_2}{x_2^2} & \frac{L_2}{y_2^2} & \frac{1}{x_2} & \frac{1}{y_2} \\ \frac{L_3}{x_3^2} & \frac{L_3}{y_3^2} & \frac{1}{x_3} & \frac{1}{y_3} \\ \frac{L_5}{x_5^2} & \frac{L_5}{y_5^2} & \frac{1}{x_5} & \frac{1}{y_5} \end{vmatrix} = 0 \quad \ldots \quad (11.1)$$

The centre circles (10.5) and (11.1) intersect at the circum-centre O_3 of the lines L_1, L_2, L_3, and in one other point, call this P_5.

The matrix common to (10.5) and (11.1) is

$$\left\|\begin{matrix} \frac{L_1}{x_1^2} & \frac{L_1}{y_1^2} & \frac{1}{x_1} & \frac{1}{y_1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{L_3}{x_3^2} & \frac{L_3}{y_3^2} & \frac{1}{x_3} & \frac{1}{y_3} \end{matrix}\right\|$$

the derminants of the third order of which vanish for O_3 [see (10.4)], and for O_3 only. It follows by the principle stated in §2, that for the other point of intersection P_5, all the fourth order determinants contained in the matrix

$$\left\|\begin{matrix} \frac{L_1}{x_1^2} & \frac{L_1}{y_1^2} & \frac{1}{x_1} & \frac{1}{y_1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{L_5}{x_5^2} & \frac{L_5}{y_5^2} & \frac{1}{x_5} & \frac{1}{y_5} \end{matrix}\right\| \quad \ldots \quad (11.2)$$

vanish. Hence we infer that in a system of five lines all the five centre circles C_4 are concurrent at a point P_5 given by the matrix (11.2).

12. In § 4, we saw that all the determinants of the fourth order contained in the matrix (4.5), namely,

$$\left\|\begin{matrix} \frac{L_1}{x_1^2}, & \frac{L_1}{x_1 y_1}, & \frac{L_1}{y_1^2}, & \frac{1}{x_1}, & \frac{1}{y_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{L_4}{x_4^2}, & \frac{L_4}{x_4 y_4}, & \frac{L_4}{y_4^2}, & \frac{1}{x_4}, & \frac{1}{y_4} \end{matrix}\right\|$$

vanished for the Miquel point M_4; One of these determinants is (11.1) denoting the centre-circle. Hence we see that the centre-circle C_4 of four lines passes through the Miquel point M_4—a well-known property.

Again, in § 5 we saw that the Miquel circle of five lines is given by

$$\begin{vmatrix} \frac{L_1}{x_1^2} & \frac{L_1}{x_1 y_1} & \frac{L_1}{y_1^2} & \frac{1}{x_1} & \frac{1}{y_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{L_5}{x_5^2} & \frac{L_5}{x_5 y_5} & \frac{L_5}{y_5^2} & \frac{1}{x_5} & \frac{1}{y_5} \end{vmatrix} = 0$$

Expanding in terms of the elements of the second column and noting that each minor determinant in the expansion vanishes for the point P_5 determined by the lines, we conclude that

in a system of five lines, the point of coucurrence, P_5, of the five centre-circles of the lines taken four at a time, lies on the Miquel circle M_5.

We believe this theorem to be new.

13. The centre, 0_4, of the centre circle, C_4, of four lines.

The centre circle of four lines (L_1, L_2, L_3, L_4) is given by

$$\begin{vmatrix} \frac{L_1}{x_1^2} & \frac{L_1}{y_1^2} & \frac{1}{x_1} & \frac{1}{y_1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{L_4}{x_4^2} & \frac{L_4}{y_4^2} & \frac{1}{x_4} & \frac{1}{y_4} \end{vmatrix} = 0$$

Its centre will be the point of intersection of the tangents to it at I and J Denoting the determinant by Δ, the tangent at I (x_1, x_2, x_3, x_4) is given by

$$x_1 \frac{d\,\Delta}{dL_1} + x_2 \frac{d\,\Delta}{d\,L_2} + x_3 \frac{d\,\Delta}{dL_3} + x_4 \frac{d\,\Delta}{d\,L_4} = 0.$$

Expanding Δ in terms of the elements of the first row, and denoting the minors of $\frac{L_1}{x_1^2}$, $\frac{L_1}{y_1^2}$ by A and B, we get

$$\Delta = A\,\frac{L_1}{x_1^2} + B\,\frac{L_1}{y_1^2} + \text{terms not containing } L_1.$$

$$\therefore \qquad x_1\,\frac{d\,\Delta}{dL_1} = \frac{A}{x_1} + \frac{Bx_1}{y_1^2}$$

$$\therefore \quad \Sigma\, x_1\,\frac{d\,\Delta}{dL_1} = \Sigma\,\frac{A}{x_1} + \Sigma\,\frac{Bx_1}{y_1^2}$$

$$= \begin{vmatrix} \frac{1}{x_1} & \frac{L_1}{y_1^2} & \frac{1}{x_1} & \frac{1}{y_1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{x_4} & \frac{L_4}{y_4^2} & \frac{1}{x_4} & \frac{1}{y_4} \end{vmatrix}$$

$$+ \begin{vmatrix} \frac{L_1}{x_1^2} & \frac{x_1}{y_1^2} & \frac{1}{x_1} & \frac{1}{y_1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{L_4}{x_4^2} & \frac{x_4}{y_4^2} & \frac{1}{x_4} & \frac{1}{y_4} \end{vmatrix}$$

The first of these vanishes. Hence the equation of the tangent at I becomes

$$\begin{vmatrix} \frac{L_1}{x_1^2} & \frac{x_1}{y_1^2} & \frac{1}{x_1} & \frac{1}{y_1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{L_4}{x_4^2} & \frac{x_4}{y_4^2} & \frac{1}{x_4} & \frac{1}{y^4} \end{vmatrix} = 0 \qquad \ldots (13.1)$$

Dividing the rows by x_1, x_3, x_3, x_4, and changing the order of columns, this can be written

$$\begin{vmatrix} \frac{L_1}{x_1^3} & \frac{1}{x_1^2} & \frac{1}{x_1 y_1} & \frac{1}{y_1^2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{L_4}{x_4^3} & \frac{1}{x_4^2} & \frac{1}{x_4 y_4} & \frac{1}{y_4^2} \end{vmatrix} = 0 \quad \ldots (13.2)$$

Similarly the tangent at J (y_1, y_2, y_3, y_4) is given by

$$\begin{vmatrix} \frac{L_1}{y_1^3} & \frac{1}{x_1^2} & \frac{1}{x_1 y_1} & \frac{1}{y_1^2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{L_4}{y_4^3} & \frac{1}{x_4^2} & \frac{1}{x_4 y_4} & \frac{1}{y_4^2} \end{vmatrix} = 0 \quad \ldots (13.3)$$

Hence applying the principle in § 2, we see that all the determinants of the fourth order contained in the matrix

$$\left\Vert \begin{matrix} \frac{L_1}{x_1^3} & \frac{L_1}{y_1^3} & \frac{1}{x_1^2} & \frac{1}{x_1 y_1} & \frac{1}{y_1^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{L_4}{x_4^3} & \frac{L_4}{y_4^3} & \frac{1}{x_4^2} & \frac{1}{x_4 y_4} & \frac{1}{y_4^2} \end{matrix} \right\Vert \quad \ldots (13.4)$$

vanish for O_4 the centre of the centre-circle; that is, this matrix represents O_4.

Taking now a fifth line L_5 we readily deduce that the determinant

$$\begin{vmatrix} \frac{L_1}{x_1^3} & \frac{L_1}{y_1^3} & \frac{1}{x_1^2} & \frac{1}{x_1 y_1} & \frac{1}{y_1^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{L_5}{x_5^3} & \frac{L_5}{y_5^3} & \frac{1}{x_5^2} & \frac{1}{x_5 y_5} & \frac{1}{y_5^2} \end{vmatrix} = 0 \quad (13.5)$$

represents a circle passing through the five centres O_4 of the centre circles of the lines L_1, L_2, ... L_5, taken in sets of four; that is, it is the equation of the centre-circle C_5.

Introducing a sixth line L_6, we see that the six centre-circles C_5 of the system L_1. L_2, ... L_6, are concurrent at a point P_6 for which all the determinants of the matrix

$$\left\| \begin{array}{ccccc} \dfrac{L_1}{x_1^3} & \dfrac{L_1}{y_1^3} & \dfrac{1}{x_1^2} & \dfrac{1}{x_1 y_1} & \dfrac{1}{y_1^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{L_6}{x_6^3} & \dfrac{L_6}{y_6^3} & \dfrac{1}{x_6^2} & \dfrac{1}{x_6 y_6} & \dfrac{1}{y_6^2} \end{array} \right\| \quad \ldots \quad (13.6)$$

vanish; that is, this matrix represents the point P_6.

14. The equation to the tangent at I $(x_1, x_2, x_3, x_4, x_5)$ to the centre circle C_5 given by (13.6.) can now be shown to be

$$\begin{vmatrix} \dfrac{L_1}{x_1^4} & \dfrac{1}{x_1^3} & \dfrac{1}{x_1^2 y_1} & \dfrac{1}{x_1 y_1^2} & \dfrac{1}{y_1^3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{L_5}{x_5^4} & \dfrac{1}{x_5^3} & \dfrac{1}{x_5^2 y_5} & \dfrac{1}{x_5 y_5^2} & \dfrac{1}{y_5^3} \end{vmatrix} = 0 \quad (14.1)$$

while the tangent at J $(y_1, y_2, \ldots y_5)$ is

$$\begin{vmatrix} \dfrac{L_1}{y_1^4} & \dfrac{1}{x_1^3} & \dfrac{1}{x_1^2 y_1} & \dfrac{1}{x_1 y_1^2} & \dfrac{1}{y_1^3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{L_5}{y_5^4} & \dfrac{1}{x_5^3} & \dfrac{1}{x_5^2 y_5} & \dfrac{1}{x_5 y_5^2} & \dfrac{1}{y_5^3} \end{vmatrix} = 0 \quad (14.2)$$

Hence, for the centre, O_5, of the circle C_5 the determinants of the fifth order contained in the matrix

$$\left\|\begin{matrix} \frac{L_1}{x_1^4} & \frac{L_1}{y_1^4} & \frac{1}{x_1^3} & \frac{1}{x_1^2 y_1} & \frac{1}{x_1 y_1^2} & \frac{1}{y_1^3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{L_5}{x_5^4} & \frac{L_5}{y_5^4} & \frac{1}{x_5^3} & \frac{1}{x_5^2 y_5} & \frac{1}{x_5 y_5^2} & \frac{1}{y_5^3} \end{matrix}\right\| \quad \ldots \quad (14.3)$$

vanish. Hence this matrix denotes O_5.

Taking now a sixth line L_6, we see that there exists a centre-circle C_6 given by the determinant

$$\begin{vmatrix} \frac{L_1}{x_1^4} & \frac{L_1}{y_1^4} & \frac{1}{x_1^3} & \frac{1}{x_1^2 y_1} & \frac{1}{x_1 y_1^2} & \frac{1}{y_1^3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{L_6}{x_6^4} & \frac{L_6}{y_6^4} & \frac{1}{x_6^3} & \frac{1}{x_6^2 y_6} & \frac{1}{x_6 y_6^2} & \frac{1}{y_6^3} \end{vmatrix} = 0 \quad (14.4)$$

15. The process can be continued for any number of lines. If we take lines $L_1, L_2, \ldots L_n$, the centre-circle C_n of the system is given by the determinant

$$\begin{vmatrix} \frac{L_1}{x_1^{n-2}} & \frac{L_1}{y_1^{n-2}} & \frac{1}{x_1^{n-3}} & \frac{1}{x_1^{n-4} y_1} & \cdots & \frac{1}{y_1^{n-3}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{L_n}{x_n^{n-2}} & \frac{L_n}{y_n^{n-2}} & \frac{1}{x_n^{n-3}} & \cdots & \cdots & \frac{1}{y_n^{n-3}} \end{vmatrix} = 0 \quad (15.1)$$

In this determinant if we suppress the row corresponding to any line L_r, the resulting matrix represents the centre O_{n-1} of the centre-circle formed by the remaining $n-1$ lines; while if we add a row, corresponding to a new line L_{n+1}, we get a matrix representing the point P_{n+1} corresponding to the system of $n+1$ lines.

"Space inside an Atom."*

BY S. V. RAMAMURTY, M.A., I.C.S.

Bohr has worked out a theory regarding the structure of an atom, The nucleus is considered as made up of a number of protons and electrons. Round this there are revolving a number of electrons. Each electron moves in a circular or elliptic path with a certain quantum number. The quantum number is simply 1, 2, 3, 4 ... attached to the 1st, 2nd, 3rd, 4th ... orbits. It is possible for an electron to move from a path with one quantum number to a path with another quantum number. But it is, under the theory, not possible for the electron to move in between two orbits with two successive quantum numbers. Under Newtonian dynamics, a particle moving in an orbit, if slightly disturbed, moves in an orbit slightly removed—the degree of removal depending on the degree of disturbance. But under the Bohrian theory, the electron if it changes its orbit at all can only jump to a new orbit with a different quantum number and not move to a slightly varied orbit. What is in between different possible orbits is not known. We still continue to speak of the inside of the atom as a three-dimensional space but it has got curious gaps about the nature of which we have no information The space inside an atom is pictured by Bohr as a discontinuous space.

Further Bohr has found a relation between the radii of the successive circular orbits possible. The associated elliptical orbits (*i.e*, elliptical orbits with the same quantum number as a given circular orbit) can be derived easily from the circular orbits. The relation is that the radii of the successive circular orbits vary as the squares of natural numbers.

An obvious explanation of the discontinuous space of Bohr is that it is some transformation of a continuous space.

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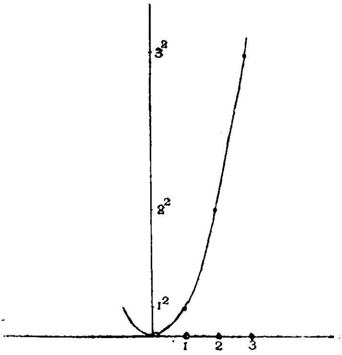
* Read at the Fifth Conference of the *Indian Mathematical Society*, Bangalore, 1926.

Take a simple analogy.

Take a simple unit and go on measuring it on the x-axis Every time it completes a measurement of itself, let a bright point of light be placed. Then we get bright points of light at 1, 2, 3, ... on the x-axis.



These points of light can be brought as near to each other as possible and the x-axis can look as nearly as we desire as a continuous line of light. Now erect perpendiculars at 1, 2, 3, ... the perpendiculars being $1^2, 2^2, 3^2 \ldots$ These points lie on a parabolic curve. Even if we mark these points on the y-axis, they lie on the y line at 1, 4. 9, 16 ... Thus if every time the unit completes a measurement of itself on the x-axis, we cause a bright point of light at the corresponding point on the parabolic curve or on the y-axis, we get a series of bright points. This series is a discontinuous series. However near we bring the points on the x-axis, the points on the curve or on y-axis are bound to be discontinuous.

Supposing the unit measuring itself on the x-axis is an atom, we have on the x-axis a continuous string of atoms. If however we cannot see the position of an atom but can see only the positions marked by the squares of the numbers of the atoms, we see an arrangement of positions which is not continuous while actually the atoms are arranged continuously.

Now coming back to the electrons, we can not actually see electrons. We do not actually peer inside an atom, we can only make statements about electrons by watching results produced by electrons—results which can be measured by our material senses or instruments. When therefore electrons are found to move in successive orbits with radii varying as $1^2 . 2^2, 3^2, \ldots$ and when we find that we know absolutely nothing of what is in between the orbits and that the electrons themselves are quite unconcerned with the in-between regions, the obvious deduction is that what we are depicting as the motion of the electrons is not the motion but a transformation of the motion just as the succession of bright points on the parabolic curve or y-axis we have referred to is not the motion of an atom which goes on measuring its length over and over again but a transformation of that motion. What may this transformation be?

In my paper on "Time, Space, Matter and Mind," published in the *Journal of the Indian Mathematical Society*, Vol. XV, Nos. 1 and 2, I have held that the number of dimensions of a continuum is the number of kinds of atoms of time, space, matter and mind that are contained in it and that a continuum of time and space particles is two-dimensional. Now inside an atom or matter, the only kinds of particles which exist are time and space particles. Therefore the space inside an atom should be on my theory two-dimensional and not three-dimensional as is now held. Further I have obtained a relation $t = r^2$ as the relation between time and space, where t and r are small elements of time and space. In a three-dimensional continuum of time, space and matter particles, let r be an element of space. Transform this to time by the relation $t = r^2$. We get a continuum of time in place of a continuum of space.

Now Bohr tells us that the radius of an atom is cut by orbits at points at distances $1^2, 2^2, 3^2 \ldots$ from the centre. Let the group of these points be transformed by the relation $t = r^2$. We then get a radius of t particles in place of a radius of r particles. The succession of discontinuous orbits get arranged into a succession of continuous orbits, Hence the transformation of space into time is just the transformation which converts a Bohrian discontinuous set of orbits in to a set of continuous orbits. Thus my statement that the inside of an atom is a continuum of 2-dimensional time instead of 3-dimensional space is corroborated by the Bohrian theory of intra-atomic orbits.

It is a myth to say that the inside of an atom is 3-dimensional. It is 2-dimensional. The space inside an atom is flat.

What is the nature of this flat continuum ?

I have shown in the previous paper referred to that in Euclidean space of 3 dimensions,

$$r^2 = x^2 + y^2$$

and in space of 4 dimensions

$$r^3 = x^3 + y^3.$$

In Euclidean (or symmetrical) space of 2 dimensions,

$$r = x + y.$$

Hence the x and y-axes are coincident lines.

If $y = 0$, $r = x$. A straight line has therefore only one direction in 2-dimensional space.

Again compare the equation

$$r = x + y$$

with the equation

$$r\,E^{\theta} = x + y$$

for 4-dimensional space.

(*Vide*: "Time, Space, Matter and Mind," *Journal of the Indian Mathematical Society*, Vol. XV, No. 2).

As the x and y-axes are coincident in 2-dimensional space, θ which by comparison with the corresponding equation for 3-dimensional space, *viz.*

$$re^{i\theta} = x + iy$$

is seen to be the angle between x and r, is zero.

Hence, $r = x + y$ for 2-dimensional space is the same as

$$r\mathrm{E}^{\theta} = x + y \text{ for 4-dimensional space.}$$

In the latter we have 3 equations to describe different parts of it, *viz.*,

$$r\mathrm{E}^{\theta} = y + x,$$

$$r\mathrm{E}^{\wedge\theta} = y \wedge x,$$

$$r\mathrm{E}^{\mathrm{A}\theta} = y \,\mathrm{A}\, x,$$

Thus 2-dimensional space is a section of 4-dimensional space.

As we proceed to lengths as large as planetary distances, our 3-dimensional space reaches the 4-dimensional space. As we proceed to lengths as small as those inside an atom, our 3-dimensional space reaches to 2-dimensional space. The latter is however a section of 4-dimensional space.

Hence it is seen that 3-dimensional space is the boundary between two regions of 4-dimensional space.

Again, motion is a quality of matter. As there is no matter inside the atom where there are only space and time particles, there is no motion inside the atom. The periodic motions perceived outside the atom apparently exist in an integrated form inside the atom.

It seems to me that the mysteries of electricity and of the quantum are involved in a study of a 2-dimensional space.

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NOTES AND QUESTIONS.

Notes and Questions.

On a certain principle in Determinants.

1. The following theorem on determinants which is assumed in the paper on "Miquel Points and Circles and Centre-circles of a System of Lines" (Vol. XVI No 12, page 271 is proved here for ready reference.

If in a matrix containing $n+1$ rows and n columns two of the determinants of the nth order contained in the matrix vanish, then all the determinants of the nth order contained in the matrix vanish—provided that in the matrix of $n-1$-rows and n columns the contained determinants of the $(n-1)$th order do not all vanish.

2. Thus in the matrix

$$M \equiv \left\| \begin{matrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ c_1 & c_2 & \cdots & c_n \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ l_1 & l_2 & \cdots & l_n \end{matrix} \right\|$$

with $n+1$ rows and n columns let it be given that the determinants obtained by suppressing the first and second rows are zero ; and that in the matrix common to them, namely in

$$\left\| \begin{matrix} c_1 & c_2 & \cdots & \cdots & c_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ l_1 & l_2 & \cdots & \cdots & l_n \end{matrix} \right\|$$

the contained determinants of the $(n-1)$th order *are not all zero*. Then all the determinants of the nth order contained in M vanish, To prove this, consider the determinant

$$\Delta \equiv \begin{vmatrix} x_1 & x_2 & \dots & x_n \\ c_1 & c_2 & \dots & c_n \\ \dots & \dots & \dots & \dots \\ l_1 & l_2 & \dots & l_n \end{vmatrix}$$

By hypothesis the minors of the terms in the first row are not all zero; let them be denoted by $X_1\ X_2\ X_3 \dots \dots X_n$. Now, Δ vanishes when for $x_1\ x_2\ x_3 \dots x_n$ we write first $a_1\ a_2\ a_3 \dots a_n$ and secondly $b_1\ b_2\ b_3 \dots b_n$. This is from hypothesis. And Δ vanishes also when for $x_1\ x_2\ x_3 \dots x_n$ we write the elements in the third, fourth, ... or $(n+1)$th rows, since the resulting determinants have two rows identical. Hence we have the following equations:—

$$\begin{aligned} a_1 X_1 + a_2 X_2 + \dots + a_n X_n &= 0 \\ b_1 X_1 + b_2 X_2 + \dots + b_n X_n &= 0 \\ c_1 X_1 + c_2 X_2 + \dots + c_n X_n &= 0 \\ \dots \quad \dots \quad \dots \quad \dots \quad &\dots \\ l_1 X_1 + l_2 X_2 + \dots + l_n X_n &= 0 \end{aligned}$$

where $X_1\ X_2 \dots \dots X_n$ are a set of quantities *which are not all zero.* By eliminating them from these equations taken n at a time, we get the property stated.

3. The result is readily perceived to be true if its geometrical significance is grasped Let us take, for example, a matrix of 5 rows and 4 columns. The elements of each row may be regarded as the homogeneous co-ordinates of a point in ordinary space. We have thus five points A, B, C, D and E whose co-ordinates are $(a_1\ a_2\ a_3\ a_4) \dots \dots (e_1\ e_2\ e_3\ e_4)$. When the determinant formed by four rows vanishes, the corresponding four points are co-planar. Thus by hypothesis the points BCDE and ACDE form two co-planar tetrads. These two planes may be the same or different. In the former case the five points are coplanar and hence every determinant of the fourth order formed from the matrix of their co-ordinates vanishes. In the latter case

the two planes have a line in common and, since C, D and E belong to both the planes, they must lie on this line. In this case the co-ordinates of any one of the three points, say of E, are the same linear combinations of the co-ordinates of C and D. In other words, we have relations of the type

$$e_1 = \lambda c_1 + \mu d_1$$

$$e_2 = \lambda c_2 + \mu d_2$$

$$e_3 = \lambda c_3 + \mu d_3$$

$$e_4 = \lambda c_4 + \mu d_4$$

and hence all the determinants of the fourth order formed out of the last three rows vanish. This is the case excluded by the hypothesis.

V. RAMASWAMI AIYAR.

Indeterminate Equation of the First Degree.

§ 1. The process of solving indeterminate equations of the first degree is simplified by a judicious use of a fairly common notation, amounting to a definition, whereby the fraction b/a denotes an integer x such

$$ax \equiv b \quad (\text{mod. } n)$$

provided a is prime to n. Fractions as thus defined have a unique value, in the sense that their equivalent integers are congruent to each other, modulus n. This definition may be extended to cases where a is not prime to n, but fractions of this type have not a unique value, and we need not discuss them further, though we will have occasion to employ them. We will call a fraction "one-valued," if the denominator is prime to the modulus, other fractions being called "many-valued."

§ 2. One-valued fractions combine according to the laws

$$b/a \pm b'/a' \equiv (a'b \pm ab')/aa' ;$$

$$(b/a) \times (b'/a') \equiv (bb'/aa') ;$$

but $(b/a) \div (b'/a')$ is *not* $a'b/ab'$, unless the latter fraction is also one valued. It is also to be noted that a many-valued fraction (gb/ga) g being a factor of n, is not congruent to the one-valued fraction (b/a).

The following results are easily seen to be true.—

(i) $b/a \equiv (b+pn)/(a+qn)$, p and q being any integers;

(ii) $b/a \equiv (kb)/(ka)$, k being an integer prime to n;

(iii) If $b/a \equiv d/c$, then each is congruent to

$$(\lambda b + \mu d)/\lambda a + \mu c)$$

where λ and μ are any integers whatever, provided this fraction is also one-valued.

§ 3. The application of the notation is best explained by a numerical example. Let it be required to solve

$$59x - 73y = 2.$$

Taking 59 to be the modulus, we have

$$73y \equiv -2$$

or $$y \equiv \frac{-2}{73} \equiv \frac{-2}{73-59} \equiv \frac{-2}{14} \equiv \frac{-1}{7} \equiv \frac{-10}{70} \equiv \frac{-2-(-10)}{73-70} \equiv \frac{8}{3} \equiv \frac{40}{15}$$

$$\equiv \frac{-2\ -40}{14\ -15} \equiv +42 = 42 + 59t$$

giving $$x = \frac{1}{59}(73y+2) = 52 + 73t.$$

It may be verified that if we take 73 to be the modulus, the smae solution is obtained; for

$$x \equiv \frac{2}{59} \equiv \frac{2}{-73+59} \equiv \frac{2}{-14} \equiv \frac{8}{-56} \equiv \frac{2+8}{59-56} \equiv \frac{10}{3} \equiv \frac{10-73}{3}$$

$$\equiv \frac{-63}{3} \equiv -21 \equiv 52 = 52 + 73t.$$

§ 4. No new principle is involved in this method, the only merit of which is its comparative simplicity when dealing with large integers. A point which must be carefully borne in mind is that though we may at any stage multiply the numerator and denominator by a number k which is not prime to the modulus, and treat the resulting fraction as if one-valued, we cannot except in special cases, cancel out a common factor f, if f is not prime to the modulus. For example, let the given equation be

$$35x - 24y = 1.$$

Taking the modulus to be 24,

$$x \equiv \frac{1}{35} \equiv \frac{1 \times 2}{35 \times 2} \equiv \frac{2}{70} \equiv \frac{2}{70 - 3 \times 24} \equiv \frac{2}{-2}.$$

One is apt to say at this stage that $x \equiv -1$, which is not correct. The fact is that in removing the common factor 2, we have implicitly divided the modulus also by 2. The correct procedure is to leave the fraction as it is, and deal with it as if it were one-valued.

$$x \equiv \frac{1}{35} \equiv \frac{2}{-2} \equiv \frac{2 \times 17}{-2 \times 17} \equiv \frac{34}{-34} \equiv \frac{1 + 34}{35 - 34} \equiv 35 \equiv 11 \equiv 11 + 24t$$

giving $$y = 16 + 35t.$$

The justification for the procedure is simple. We leave it to the reader to find it out.

BALAKRAM.

On the Reduction of the General Equation of the Second Degree.

In text-books on 'Conic Sections,' the equation of the second degree is reduced to its normal form by change of origin and axes. In the following note the transformation is effected algebraically, the method depending on the application of the following elementary identities :—

$$(\alpha) \quad (a^2 + b^2)(l^2 + m^2) = (al + bm)^2 + (am - bl)^2,$$

$$(\beta) \quad (a^2 - b^2)(l^2 - m^2) = (al - bm)^2 - (am - bl)^2,$$

and $$(\gamma) \quad 4ab \, . \, lm = (al + bm)^2 - (al - bm)^2.$$

Let the equation be as usual

$$\phi(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Now suppose one of the quantities a, b, say a, is not zero; then

$$a \cdot \phi(xy) = a^2x^2 + 2ax(hy+g) + aby^2 + 2afy + ac$$
$$= (ax + hy + g)^2 + Cy^2 - 2Fy + B$$
$$= (ax + hy + g)^2 + C^{-1}(Cy - F)^2 + a \Delta C^{-1}. \quad \ldots \quad (1)$$

where B, C, F are the minors of the corresponding elements in

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Case 1. $C > 0$. Multiply (1) by $C(l^2 + m^2)$ and apply identity (α), we have

$$a(l^2 + m^2) C \phi(xy) = L^2 + M^2 + a(l^2 + m^2) \Delta. \quad \ldots \quad (2)$$

where

$$L = l\sqrt{C}.(ax + hy + g) + m(Cy - F)$$

and

$$M = m\sqrt{C}.(ax + hy + g) - l(Cy - F)$$

It is clear form (2) that $L = 0$ and $M = 0$ are the equations to a pair of conjugate diameters of the ellipse for all values of $l : m$, the axes being inclined at any angle whatever.

Now choosing $l : m$ so that the straight lines $L = 0$ and $M = 0$ are at right angles we reduce the equation to the normal form

$$\frac{X^2}{A^2} + \frac{Y^2}{B^2} = 1$$

where $X = 0$, $Y = 0$ are principal axes of the conic.

Case 2. $C < 0$. By using identity (β) we may reduce the equation in this case in the same way as above. If however we have no objection to the use of imaginary quantities, this case can be brought under Case 1. This may be seen by observing that (β) can be deduced from (α) by putting mi and bi for m, b in (α).

Case 3. $C = 0$. The method for this case is given in all standard works and need not be repeated here. (*Vide* Askwith: *Analytical Geometry of the Conic Sections.*, § 155, page 143).

Case 4. $a = b = 0,\ h \neq 0$. In this case

$$h\,\phi(x, y) = 2h^2xy + 2hgx + 2hfy + ch$$
$$= 2(hx + f)(hy + g) + ch - 2fg. \quad \dots \quad (3)$$

Now multiply (3) by $2\,lm$ and use identity (γ); we then have

$$2\,lm \, . \, h\,\phi(x, y) = L^2 - M^2 + 2lm\,(ch - 2fg)$$

where

$$L \equiv l\,(hx + f) + m\,(hy + g)$$

and

$$M \equiv l\,(hx + f) - m\,(hy + g)$$

It is clear as before that $L = 0$ and $M = 0$ are a pair of conjugate diameters for all values of $l : m$, The final step may be taken as in Case 1.

The method employed in Case 4 may be extended to the first two cases. Thus the whole problem may be made to depend on the use of (γ).

The method given above is perhaps longer than the one usually given in text-book; it has however the merit of being a purely algebraic reduction independent of geometric considerations.

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Solutions.

Question 1279.

(G. V. GURJAR):—Let A and B be two point charges at points $(-a, 0)$ and $(a, 0)$. The equation of the lines of force in the plane can be found from

$$\frac{dy}{dx} = \frac{y}{x + a\,\dfrac{PB^3 - PA^3}{PB^3 + PA^3}}$$

where P is the point (x, y).

This equation admits of a solution in the form

$$\frac{x+a}{PA} + \frac{x-a}{PB} = \text{constant.}$$

Explain the method of solution.

Solution by S L. Malurkar and K. Satyanarayana.

Take the case of a number of point charges in a straight line at points $(a_s, 0)$ and of magnitude e_s. ($s = 1, 2, \ldots$).

The potential at a point P (x, y) is

$$V = \sum_s \frac{e_s}{r_s} \qquad \text{where} \qquad r_s^2 = \overline{x - a_s}^2 + y^2$$

The lines of force being orthogonal to the curves V = const. are given by

$$\frac{dy}{dx} = \frac{\partial V}{\partial y} \Big/ \frac{\partial V}{\partial x} = \frac{\sum_s \dfrac{e_s}{r_s^2} \cdot \dfrac{\partial r_s}{\partial y}}{\sum_s \dfrac{e_s}{r_s^2} \cdot \dfrac{\partial r_s}{dx}} .$$

$$= \frac{\sum_s \dfrac{e_s}{r_s^2} \cdot \dfrac{y}{r_s}}{\sum_s \dfrac{e_s}{r_s^2} \cdot \dfrac{(x - a_s)}{r_s}}$$

Taking $s = 2$, $e_1 = e_2$, we obtain the differential equation given in the question.

$$\therefore \frac{dy}{dx} = \frac{\sum \frac{e_s}{r_s^2} \cdot \frac{y^3}{r}}{\sum_s \frac{e_s}{r_s^2} \cdot \frac{y(x-a_s)}{r_s}}$$

$$= \frac{\sum_s \frac{e_s (r_s^2 - \overline{x-a_s}^2)}{r_s^2}}{\sum_s \frac{e_s (x-a_s)}{r_s^2} \cdot \frac{y}{r_s}}$$

$$= \frac{\sum \frac{e_s}{r_s} - \sum \frac{e_s \cdot (x-a_s)}{r_s^2} \cdot \frac{(x-a_s)}{r_s}}{\sum \frac{e_s (x-a_s)}{r_s^2} \cdot \frac{y}{r_s}}$$

$$= \frac{\frac{\partial}{\partial x}\left\{ \sum \frac{e_s (x-a_s)}{r_s} \right\}}{-\frac{\partial}{\partial y}\left\{ \sum \frac{e_s (x-a_s)}{r_s} \right\}}$$

The equation being exact, the lines of force are given are given by

$$\sum \frac{e_s (x-a_s)}{r_s} = \text{const.}$$

Question 1425.

(B. B. Bagi) :—If R, R_1, R_2, R_3 are the circum-radii of the four similar triangles whose sides touch a circle of radius r, show that

(1) $R_1 + R_2 + R_3 = R$;

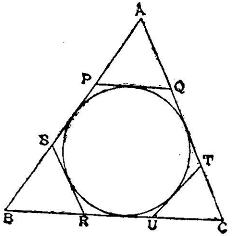
(2) $R_1^{-1} + R_2^{-1} + R_3^{-1} - R^{-1} = 4r^{-1}$;

and (3) that the angles of the triangles are 2α, 2β, 2γ

where $$\tan \alpha = \left\{ \frac{R_1^{-1} - R^{-1}}{R_2^{-1} + R_3^{-1}} \right\}^{\frac{1}{2}}, \text{ etc.}$$

Solutions by N.P. Subramanian and G. V. Krishnaswamy

If PQ, RS and TU be the tangents parallel to BC, CA, AB respectively to the in-circle of the triangle ABC, then ABC, APQ, BRS and CTU are the four similar triangles touching the circle I of radius r. Let r_1, r_2, and r_3 be the ex-radii of the triangle ABC.



Since in similar triangles corresponding lengths are proportional, we have

$$\frac{R_1}{R} = \frac{r}{r_1}$$

for r is the ex-radius corresponding to the $\angle A$ of the triangle APQ.

Hence $$R : R_1 : R_2 : R_3 = \frac{1}{r} : \frac{1}{r_1} : \frac{1}{r_2} : \frac{1}{r_3}.$$

The relations (1) and (2) follow directly from

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$$

and $$r_1 + r_2 + r_3 - r = 4R.$$

Lastly if a, b, c be the sides of a triangle

$$\tan^2 \frac{A}{2} = \frac{(s-b)(s-c)}{s(s-a)} = \frac{\dfrac{1}{(s-a)} - \dfrac{1}{s}}{\dfrac{1}{(s-b)} + \dfrac{1}{(s-c)}}$$

$$= \frac{r_1 - r}{r_2 + r_3} = \frac{R_1^{-1} - R^{-1}}{R_2^{-1} + R_3^{-1}}$$

Since the angles of the triangles are taken as 2α, 2β and 2γ

$$\tan\alpha : \tan\beta : \tan\gamma = \frac{R_1^{-1} - R^{-1}}{R_2^{-1} + R_3^{-1}} : \frac{R_2^{-1} - R^{-1}}{R_3^{-1} + R_1^{-1}} : \frac{R_3^{-1} - R^{-1}}{R_1^{-1} + R_2^{-1}}.$$

It should be noted that these relations will hold only if R_1, R_2, R_3, R are taken so as to correspond to r_1, r_2, r_3, r, *i.e.*, as the circum-radii of the triangles APQ, BSR, CTU and ABC respectively. Otherwise they will have to be replaced by others in which the R's are interchanged.

Other Solutions by V. A. Mahalingam and B. Achyutaram Sastri.

Question 1433.

(V. RAMASWAMY AIYER AND M. BHIMASENA RAO) :—Two quadrangles ABCD, A′B′C′D′ are such that six sets of points like (BC A′D′) are orthocentric. Prove that the six meets of their corresponding sides (like BC, B′C′) and the six meets of their non-corresponding sides (like BC, A′D′) form a set of twelve points lying on a circle.

Show that the quadrangles are fixed when three vertices of the one and the non-corresponding vertex of the other (as A, B, C, D′) are given; and show also how, when one of these quadrangles is given, the other can be found.

Solution by A. A. Krishnaswami Aiyengar and T. R. Raghava Sastry.

We shall first prove that such quadrangles exist.

Let ABCD′ be any four points and let A′, B′, C′ and D be the orthocentres of the triangles BCD′, CAD′, ABD′ and A′B′C respectively. A rectangular hyperbola passes through these eight points; for the rectangular hyperbola through ABCD′ passes through the orthocentres of the above triangles as well. Hence denoting the co-ordinates of A, B, C, D by $(ka, k/a)$ $(kb, k/b)$, etc. and the co-ordinates of A′, B′, C′, D′ by $(ka', k/a')$ ... ... etc. on the rectangular hyperbola $xy = k^2$, and noting that the co-ordinates of A′, the orthocentre of BCD′ are $\left(-k\,bcd', -\frac{k}{bcd'}\right)$ with similar expressions for the co-ordinates of B′, C′ and D, have the relations :—

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{d}{d'} = -abcd' \quad \text{... (1)}$$

We easily see from (1) that

$$ab = -\frac{1}{c'd'},\ bc = -\frac{1}{a'd'}\ \dots\ \text{etc.}$$

Also from (1) we get easily $abcd \cdot a'b'c'd' = +1$ whence we have also

$$a'b' = -\frac{1}{cd},\ b'c' = -\frac{1}{ad}\ \dots\ \dots\ \text{etc.}$$

Hence each of the 6 set of points like $abc'd'$ is orthocentric.

Now the equation of the chord BC is

$$x + bc\,y - k\,(b + c) = 0.$$

Hence the equation of any conic passing through the intersections of BC and B′C′, BC and A′D′, B′C′ and AD, A′D′ and AD is

$$\{x + bcy - k\,(b + c)\}\ \{x + ady - k\,(a + d)\}$$
$$+ \lambda\,\{x + b'c'y - k\,(b' + c')\}\ \{x + a'd'y - k\,(a' + d')\} = 0.$$

But since $b'c' = -\frac{1}{bc}$ and $a'd' = -\frac{1}{ad}$, the above equation represents a circle if $\lambda = abcd$. The equation of this circle is

$$(1 + abcd)\,(x^2 + y^2) - kx\,\{\Sigma\ a + abcd\ \Sigma\ a'\}$$
$$- ky\,\{\Sigma\ abc + abcd\ \Sigma\ a'b'c'\} = 0$$

From the symmetry of the equation it is evident that it passes the other eight points mentioned in the question.

When ABCD are given, if L is the orthocentre of ABC, its parameter is $-\frac{1}{abc}$, and the equation of DL is

$$x - \frac{d}{abc}\,y = k\left(d - \frac{1}{abc}\right).$$

But from (1) d'^2 is equal to $-\frac{d}{abc}$. Hence DL is parallel to $x + d'^2y = 0$. Since the equation of the tangent at D′ is $x + d'^2y = 2kd'$ the construction for finding D′ is as follows :—

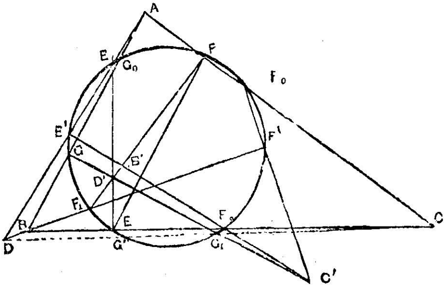
Join L, the orthocentre of ABC to D and draw the tangent paralle to DL to the rectangular hyperbola passing through ABCD. Then the point of contact is D′.

The other points can then be easily determined. From (1) $abcd = -(abcd')^2$. Hence the construction is real if one of the quantities a, b, c, d is opposite in sign to each of the others, (*i.e.*) if one of the points lies on one branch of the rectangular hyperbola and the other three on the other branch.

The equation (1) continues to hold when the signs of $abcd$ or of $a'b'c'd'$ are simultaneously changed. Hence there exist pairs of associated quadrangles ($ABCD$, $A_1B_1C_1D_1$); ($A'B'C'D'$, $A'_1B'_1C'_1D'_1$) of which any one of the first pair and any one of the second pair have the relation mentioned in the question.

We can prove that the six pairs of points like (BC, $B'C'$) and (BC, $A'D'$) lie on a circle by elementary geometry as follows:

Let ABC be any triangle, D' any point in its plane and the pedal circle of D' w.r.t. the triangle ABC cut the sides BC, CA, AB in the



pairs of points (E, E_0), (F, F_0), (G, G_0) as in the figure, where E, F, G are the feet of the perpendiculars from D' on the sides.

Let ED', FD' and GD' cut the pedal circle again at E_1, F_1, G_1, respectively; also let AE_1, BF_1, CG_1, cut the circle again at E', F', G' respectively.

Join E_0E' and let it cut $D'F$, $D'G$ (produced, if necessary) at B', C' respectively.

We now proceed to show that ($ACB'D'$) and ($ABC'D'$) are orthocentric quadrangles.

The quadrilateral $B'E'AF$ is cyclic since $B'\hat{E}'A = E_0\hat{E}E_1$ and $B'\hat{F}A$ are right angles.

Hence $B'\hat{A}F = B'\hat{E}'F = F\hat{E}E_0 = F\hat{D}'C$ = compliment of $F\hat{C}D'$. AB' is thus perpendicular to CD' and by construction $B'D'$ is perpendicular to AC. Hence $ACB'D'$ is orthocentric.

Similarly, $C'\hat{A}G = C'\hat{E}'G = G\hat{E}B = G\hat{D}'B$ = compliment of $G\hat{B}D$,

$\therefore$ AC' is perpendicular to BD' and by construction AB is perpendicular to $C'D'$. Hence $ABC'D'$ are orthocentric.

Thus, E_0E' passes through $B'C'$ the orthocentres of the triangles $D'AC$ and $D'AB$.

Similarly, we can prove that F_0F' and G_0G' pass respectively through the pairs of points (C', A') and (A', B') where A' * is the orthocentre of the triangle $D'BC$.

Since AE_1, BF_1, CG_1, are respectively perpendicular to the sides $B'C'$, $C'A'$, $A'B'$ and the perpendicuiars from A', B', C' on the sides BC, CA, AB meet at D', the straight lines, AE_1, BF_1, and CG_1 must also be concurrent at (say) D.

It is readily verified from the figure that the pedal circle of D' with respect to the triangle ABC is also the pedal-circle of D with respect to the triangle $A'B'C'$, and that the triangle ABC can be derived from the triangle $A'B'C'$ with the help of the perpendiculars from D to the sides exactly in the same way as the triangle $A'B'C'$ has been obtained from the triangle ABC with the help of the perpendiculars from D' on the sides.

Hence, we may infer that A, B, C are the orthocentres of the triangles $DB'C'$, $DC'A'$, and $DA'B'$ respectively for the same reason that A', B', C' are the orthocentres of the triangles $D'B,C$, $D'CA$, $D'AB$.

Thus the two quadrangles ABCD and $A'B'C'D'$ are associated in the manner described in the question and the twelve points (E, E_0, E_1, E'), (F, F_0, F_1, F'), $(G\ G_0, G_1, G')$ all lie on the same circle.

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* A' is not marked in the figure.

Questions for Solution

Proposers of Questions are requested to send their own solutions along with their questions.

1455. (K. V. VEDANTAM):—The director circle of the maximum inscribed ellipse of a triangle is co-axial with its polar, circum-, and nine-point circles.

1456. (V. RAMASWAMI AIYAR, M.A.):—Given three directly similar figures F_1, F_2, F_3:

(1) If they do not have a common double point, show that if ABC be any triangle homothetic to the invariable triangle, the homothetic centre being the director point, then, the triangle $\alpha\beta\gamma$ formed by any set of corresponding points of the figures is in perspective with ABC;

(2) If the figures have a common double point O, show that another system of three directly similar figures F_1', F_2', F_3', having O for a common double point, exists, such that the triangle $\alpha\beta\gamma$ formed by any set of corresponding points of F_1, F_2, F_3 is in perspective with the triangle $\alpha'\beta'\gamma'$ formed by any set of corresponding points of F_1', F_2', F_3'.

1457. (A. A. KRISHNASWAMY AYYANGAR):—Show that the trilinear equation of the Pascal line of the hexagon $AC'BA'CB'$ inscribed in a circle is of the form

$$AA'\,B'C'.\alpha + BB'.C'A'.\beta + CC'.A'B'.\gamma = 0$$

the triangle ABC being taken as the triangle of reference.

Hence deduce Steiner's and Kirkman's theorems on Pascal lines.

1458. (A. A. KRISHNASWAMI AYYANGAR):—If any transversal DEF cut the sides BC, CA, AB of a triangle at D, E, F respectively so that

$$BC.BD + CA.CE + AB.AF$$

is constant (the segments of the sides being measured positively the same way round); show that the straight line DEF envelopes a fixed tricusp inscribed in the triangle ABC.

1459. (S. D. CHOWLA):—Prove that

$$\left(\frac{2K}{\pi}\right)^2 = 1 + \left(\frac{1}{2}\right)^3 (2kk')^2 + \left(\frac{1.3}{2.4}\right)^3 (2kk')^4 + \left(\frac{1.3.5}{2.4.6}\right)^3 (2kk')^6 + \dots \dots$$

where $$K = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}}.$$

Find series for $\left(\frac{2K}{\pi}\right)^3$ and $\left(\frac{2K}{\pi}\right)^4$.

1460. (S. D. CHOWLA):—Prove that

(i) $\sigma_3 (2n-1) = \sigma_1 (2n-1) + 8 \{ \sigma_1 (1) S_1 (2n-2) + S_1 (2) \sigma_1 (2n-3) + \sigma_1 (3) S_1 (2n-4) + \dots \dots$

where $\sigma_r (n)$ denoting sum of rth powers of divisors of n and

$$S (n) = \Sigma\, \delta d^r$$

where $\delta = +1$ if d is odd, and is equal to $(-1)^{\frac{n}{d}-1}$ if d is even.

e.g., If $2n-1 = 7$, $7^3 + 1^3 = (7+1) + 8 \{ 1.12 + 3.6 + 4.3 \}$

(ii) $\phi_3 (n) = S_1 (n) + 8 \; [\sigma_1 (1) \sigma_1 (n-1) + S_1 (2) S_1 (n-2) + \sigma_1 (3) \sigma_1 (n-3) + S_1 (4) S_1 (n-4) + \dots] + 4 \left\{ S_1 \left(\frac{n}{2}\right) \right\}^2$

where n is even, and $\phi_3 (n)$ = sum of cubes of even divisors of n *minus* the sum of cubes of odd divisors of n.

e.g., If $n = 8$,

$$8^3 + 4^3 + 2^3 - 1^3 = 3 + 8 [1.8 + 3.12 + 4.6] + 4.3^2.$$

1461. (T. R. RAGHAVA SASTRY):—Four straight lines touch a circle of radius R. If r_1, r_2, r_3 and r_4 are the radii of the circum-circles of the four triangles that can be formed by the four lines and r the radius of the circle passing through their circum-centres, show that

(i) $a \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} \right) = \frac{R}{r}.$

(ii) The distance between the focus of the parabola touching the given four lines and the centre of the circle touching the same lines is $\sqrt{4ar}$, where $4a$ is the *latus rectum* of the parabola

LIST OF BOOKS AND JOURNALS RECEIVED

during the months of November and December 1926.

Journals.

1 Acta Mathematica, **45**, **46**, **47**, **48**, **49**, 1 & 2. (2 copies).
2 American Mathematical Monthly, **33**, 7 & 8.
3 Annals of Mathematics, **24**, 4.
4 Annalles de L'Ecole Normale Superieure, Nos. 6, 7, 8, 9, 10, [11.
5 Astrophysical Journal, **64**, 2.
6 Bulletin des Sciences Mathematiques, October 1926.
7 Bulletin of the Calcutta Mathematical Society, **17**, 4.
8 Contribucion al Estudio De Las Ciencias Fisicas y Mathematicas, **3**, 3.
9 Journal de Mathematique, **5**, 3.
10 Mathematische Annalen, **96**, 3 & 4.
11 Mathematical Gazette, **13**, 184. (3 copies).
12 Messenger of Mathematics, **56**, 5. (2 copies).
13 Monthly Notices of the Royal Astronomical Society, **86**, 9.
14 Nature, **118**, 2958 to 2973.
15 Philosophical Magazine, **22**, 10, 11 & 12.
16 Popular Astronomy, **34**, 8 & 9. (3 copies).
17 Proceedings of the Royal Society, **112**, 762. **113**, 763 & 764.
18 Transactions of the American Mathematical Society, **28**, 3.
19 Transactions of the Royal Society, **226**, 638.

Books

(*Presented by Prof. Hardy*).

1. Course of Pure Mathematics.
2. Integration of Functions of a Single Variable.
3. The General Theory of Dirichlet's Series.
4. Orders of Infinity.

Numbers in black type refer to the volumes, and those in ordinary type to the numbers of the issues.

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